Dupuy, presented to Hermite in the name of the President of the Republic the insignia of Grand Officer of the Legion of Honor, and the messages were read of those who from various parts of the world associated themselves with the splendid ceremony.

High testimony of admiration and sympathy was offered the great geometer more recently upon the occasion of the meeting at Paris, last August, of the international congress of mathematicians. The Congress sent him a telegram of admiration and sympathy (he was at Saint-Jean-de Luz). This act caused vast satisfaction and profound emotion to the scientist, as he wrote me in one of his last letters.

Hermite retained to the last day of his life his privileged intelligence; but his body suffered. In a long letter of his, a few days before his death, he complained of his attacks of asthma and of the lack of appetite and of sleep: he seemed to forsee the nearness of his end, so that sending me one of his works he said that this would be without doubt *the last*! and that he had in great part accomplished it at Saint-Jean-de Luz, whereby the benefit of the mild climate had reawakened his mathematical activity. This last work is a letter to Professor Pincherle published in tomo V of the "*Annali di Matematica.*" He told us also that he had sent a brief article to the new journal "Le Matematiche" of Prof. Alasia.

We will end by expressing a wish. We wish that those who have the authority would take the initiative toward an international subscription for a work containing an extended biography of the ever memorable geometer, and a minute analysis of his works; perhaps might be added some brief articles by very illustrious living mathematicians; something, in fine, which would be as a funeral crown offered to the memory of the great dead.

[Written by Juan J. Durán-Loriga for *Le Matematiche*, and translated by the English editor, G. B. Halsted.]

A PROBLEM AND ITS SOLUTION.

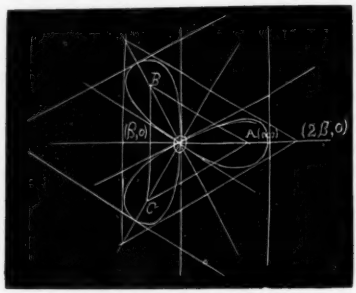
By EUPLIO CONOSCENTE, B. Sc., Math. D., Member of Circolo Mathematico di Palermo, New York City.

Find that one of these curves $(x^2+y^2)^3=a^3(x^3-3xy^2)$.

I. (a) is the locus of such points that the product of their distances from the vertices of a fixed equilateral triangle is equal to the semi-parameter. (b) The bitangents each touching the curve in two real distinct points are parallel to the sides of the fixed triangle and their six points of contact are on a circle. (c) The sides of the equilateral triangle obtained by the tangents each touching the curve in its three farthest real points from the origin of coördinates are parallel to the tangents of its real triple points. (d) Some other property showing the form of this curve.

II. (a) The "deficiency" (genus) of the curve is $p=1$ and it has three triple points and their tangents are all distinct. (b) Its arc is expressed by the Abelian integral of the first kind attached to its equation (saving a constant). (c) x and y are then two elliptic functions of this arc.

SOLUTION.



I. (a) Let the origin of rectangular coordinates, O, be the center of an equilateral triangle; one of its sides, BC, parallel to y-axis; α, being the arithmetical value of $\frac{\sqrt[3]{a^3}}{2}$ the radius of its circumscribed circle $x^2+y^2=\alpha^2$. Then its vertices are $(x=\alpha,\ y=0)$, $(x=-\frac{1}{2}\alpha,\ y=\frac{\sqrt{3}}{2}\alpha), (x=-\frac{1}{2}\alpha, y=\frac{\sqrt{3}}{2}\alpha)$ (1).

The locus of such points that the products of their distances from the points (1) is $\frac{1}{2}a^3$ is expressed by the equation

$$\sqrt{\{[(x-\alpha)^2+y^2][(x+\tfrac{1}{2}\alpha)^2+(y-\tfrac{\sqrt{3}}{2}\alpha)^2][(x+\tfrac{1}{2}\alpha)^2+(y+\tfrac{\sqrt{3}}{2}\alpha)^2]\}}=\alpha^3.$$

Developing and reducing this equation we get

$$(x^2+y^2)^3=\alpha^3(x^3-3xy^2)\ldots.(2).$$

The sides of the fixed triangle are

$$x+\alpha=0,\quad x-\sqrt{3}y-2\alpha=0,\quad x+\sqrt{3}y-2\alpha=0,\quad (\alpha^3=\tfrac{1}{2}a^3)\ldots.(3).$$

(b) These three straight lines

$$x+\beta=0,\quad x-\sqrt{3}y-2\beta=0,\quad x+\sqrt{3}y-2\beta=0,\quad (\beta^3=\tfrac{1}{4}\alpha^3)\ldots(4),$$

are bitangent to the curve respectively each in one of the following three couples of points:

$$(x=-\beta,\ y=\pm\beta),$$

$$(x=-\frac{\sqrt{3}-1}{2}\beta,\ y=-\frac{\sqrt{3}+1}{2}\beta),\ (x=\frac{\sqrt{3}+1}{2}\beta,\ y=-\frac{\sqrt{3}-1}{2}\beta)\ldots(5).$$

$$(x=-\frac{\sqrt{3}-1}{2}\beta, y=\frac{\sqrt{3}+1}{2}\beta),\ (x=\frac{\sqrt{3}+1}{2}\beta,\ y=\frac{\sqrt{3}-1}{2}\beta).$$

The bitangents (4) intersect each other in three points which form the vertices of an equilateral triangle:

$$(x=2\beta,\ y=0),\quad (x=-\beta,\ y=-\sqrt{3}\beta),\quad (x=-\beta,\ y=\sqrt{3}\beta)\ldots.(6),$$

and it is evident that the straights (3) and (4) are three couples of parallels.

All six points (5) stay on the circle $x^2+y^2=2\beta^2$.

(*c*) The circle $x^2+y^2=a^2$ *touches* the curve in its real farthest points from the origin of coördinates

$$(x=a,\ y=0),\quad (x=-\tfrac{1}{2}a,\ y=\frac{\sqrt{3}}{2}a),\quad (x=-\tfrac{1}{2}a,\ y=-\frac{\sqrt{3}}{2}a)\ldots.(7).$$

Indeed these points are the limits of three couples of real points obtained through intersection with a circle, center O and radius r, variable from 0 to a, for if $r>a$, all twelve points of intersection would be imaginary.

The tangents which touch both this circle and curve are

$$x-a=0,\quad x-\sqrt{3}y+2a=0,\quad x+\sqrt{3}y+2a=0\ldots\ (8).$$

The origin of coördinates is a real triple point of the curve (we will prove later it is alone) for the first and second partial derivatives of (2) vanish for $x=0$, $y=0$; its tangents are three distinct ones

$$x=0,\quad x-\sqrt{3}y=0,\quad x+\sqrt{3}y=0\ldots.(9),$$

which are parallel to the straights (8).

(*d*) The straight lines (3), (4), (8), and (9) give three sets of four parallel and their three directions are inclined 90°, 60°, and 30° to x-axis. The x-axis is an axis of symmetry of the curve for the degree of y in (2) is always a multiple of 2; on this straight lie the first points of (1), (6), and (7). Through two successive rotations (120°) of the coördinate system we get that the straights containing respectively the second and third points (1), (6), and (7) are also axes of symmetry and there is no other one. Then we can say that the curve is a three-lobed one, each branch has finished values and one of them is within the following straights, *i. e.*:

$$x-\sqrt{3}y=0,\quad x+\sqrt{3}y-2\beta=0,\quad x=a,\quad x-\sqrt{3}y-2\beta=0,\quad x+\sqrt{3}y=0,$$

by their points of contact and some other one we can get, it is easy to obtain the form of the curve.

II. (*a*) The curve $x'^3-3x'y'^2=1/a^3$ is the transformed one of (2) through reciprocal vectorial rays and its genus or deficiency is $p=1$; then also the first curve's genus is $p=1$, this transformation being a birational one. The curve (2) has three triple points: the origin of coördinates (a real one) and the two cyclic points [(1, i, 0) and (1, $-i$, 0)] and no more multiple points for $p=1$. The tangents of the real triple points are expressed by the equations (8) and those of the imaginary points are $(x\pm iy)^3=\frac{1}{2}a^3\ z^3$, which are six distinct tangents, of superior inflexion, for each branch is linear and of the third class.

(*b*) The equation of the curves (2), if written in polar coördinates, is

$$\rho^3 = a^3 \cos 3\theta \ldots . (10).$$

The differential expression of its arc s is

$$ds = \frac{a^3 d\theta}{\rho^2} \ldots . (11),$$

or, in Cartesian coördinates,

$$ds = \frac{a^3}{x^2+y^2} d \text{ arc tg } \frac{y}{x},$$

which is a rational function of x and y, and then $\int ds$ is an Abelian integral attached to the equation (2).

From (10) we get $\frac{d\theta}{\rho^2} = \frac{d\rho}{a^3 \sin 3\theta}$, then $s = \int \frac{a^3 d\rho}{\sqrt{(a^6 - \rho^6)}}$, which has a finished value anywhere and thus it is an Abelian integral of the first kind. But the genus of the curve is $p=1$ and it must have only an integral of the first kind attached to its equation, and then, saving a constant, this integral expresses its arc s.

Let us put $\rho^2 = 1/r$; we have

$$s = \int \frac{dr}{\sqrt{[4r^3 - (4/a^6)]}} \ldots (12),$$

which is an elliptic integral of the first kind.

The well known fundamental relation of Weierstrass between his function pu and its first derivative $p'u$, $p'^2 u = 4p^3 u - g_2 pu - g_3$, will be here

$$p'^2 s = 4p^3 s - 4/a^6 \ldots . (13).$$

The coördinates x and y of the curve (2) are expressed by the following functions

$$x = -\frac{3}{a^3} \frac{ps}{p^3 s - (4/a^6)}, \quad y = \frac{\sqrt{-3}}{2} \frac{ps\, p's}{p^3 s - (4/a^6)} \ldots . (14),$$

which are two elliptic functions of the arc s. Indeed eliminating ps and $p's$ between (13) and (14) we get

$$(x^2+y^2)^3 = a^3(x^3 - 3xy^2) = \left(\frac{-3p^2 s}{p^3 s - (4/a^6)}\right)^3.$$

It ought to be so for the coördinates of an elliptic curve are expressed by elliptic functions of the Abelian integral of the first kind relative to its equation.

ON THE CONCEPTS OF NUMBER AND GROUP.

By DR. G. A. MILLER, Cornell University, Ithica, N. Y.

There is a marked difference between the methods generally employed in the teaching of elementary geometry and arithmetic. In the former it is customary to begin by assuming that all the elements—points, lines, and planes—exist. By means of a sufficient number of postulates (axioms) these elements are then combined in such a way as to lead to a large number of interesting and valuable consequences. To complete the theory it is only necessary to prove that the postulates do not lead to any contradictory results and that they are sufficient to prove all geometric theorems.

In the theory of arithmetic we generally employ a widely different method. To develop the fundamental concept (number) we begin with the positive integers. This concept is gradually extended by the introduction of fractional, negative, irrational, and complex numbers. The last class may be regarded as pairs of real numbers. This method has been called the *genetic* method* because the general concept of number is reached by the gradual extension of the more elementary concept. The method described in the preceding paragraph may be distinguished as the *axiomatic* method.

While the genetic method seems to be the most suitable for the teaching of the elements of arithmetic it is doubtful whether it is so well suited for a logical presentation of the theory of this subject. At any rate it is a matter of great interest to develop the theory of arithmetic by the axiomatic method. The axioms of arithmetic have recently received considerable attention. The works of Dedekind, Burali-Forti, Padoa, Pieri, Peano, and others, along this line have lead to many interesting results. The last volume of th German Mathematical Association contains a paper by Professor Hilbert in which the concept of real numbers is presented according to the axiomatic method as follows:†

There is a system of elements (things) which are called numbers and they are represented by a, b, c, ... Their relations to each other and their laws of combination are completely defined by the following axioms:

A. Axioms of Combination.

1. By adding a and b we obtain a definite number c, in symbols:

$$a+b=c \text{ or } c=a+b$$

2. If a and b are given there is always one and only one number x and also one and only one number y such that

$$a+x=b, \quad y+a=b$$

*Hilbert, Jahresbericht der Deutschen Mathematik Vereinigung, 1900, page 180.
†Loc. cit.

3. There is a definite number, viz. 0, such that every number a satisfies the equations

$$a+0=a \text{ and } 0+a=a$$

4. According to a second law of combination, called multiplication, we obtain a definite number c from a and b, in symbols

$$ab=c \text{ or } c=ab$$

5. When a and b are any given numbers, except a cannot be 0, there is one and only one number x and also one and only one number y such that

$$ax=b, \quad ya=b$$

6. There is a definite number, viz. 1, such that every number a satisfies the equations

$$a.1=a \text{ and } 1.a=a$$

B. Axioms of Operation.

The following formulas apply to any three numbers; a, b, c.

1. $a+(b+c)=(a+b)+c$.
2. $a+b=b+a$.
3. $a(bc)=(ab)c$.
4. $a(b+c)=ab+ac$.
5. $(a+b)c=ac+bc$.
6. $ab=ba$.

C. Axioms of Arrangement.

1. If a and b are two different numbers then one (say a) of them must be larger than the other (b). In this case b is said to be smaller than a, in symbols $a>b$ and $b<a$.

2. From $a>b$ and $b>c$ it follows that $a>c$.

3. When $a>b$ the following inequalities must hold $a+c>b+c$ and $c+a>c+b$.

4. From $a>b$ and $c>0$ it follows that $ac>bc$ and $ca>cb$.

D. Axioms of Continuity.

1. If a and b are any two numbers such that $a>0$ and $b>0$ it is always possible to add a to itself a sufficient number of times so that the sum is greater than b; *i. e.*

$$a+a+\ldots+a>b$$

2. The numbers $a, b, c, \ldots$ form a system which is completely defined by the preceding axioms; *i. e.* it is impossible to find a different system of elements which obey all of these conditions.

Professor Hilbert observed that some of these axioms can be deduced from the others—in particular, it is easy to prove that the existence of 0 is a consequence of A 1, 2 and B 1, 2. From A 1, 2 it follows that for every number a there is a number x such that

$$a+x=a \text{ or } b+(a+x)=b+a$$

and that $a+b$ may represent any number while a is fixed. From B 1 we observe that $b+(a+x)=(b+a)+x$; *i. e.* this number x is independent of a. Finally, we conclude from B 2 that this number x is the same in

$$a+x=a \text{ and } x+a=a$$

This proves that axiom A 3 is a consequence of A 1, 2 and B 1, 2.

Most of the thoughts expressed above may be found in the article by Professor Hilbert to which reference has been given. As the thoughts seem likely to interest teachers of mathematics and as the journal in which they appeared is probably inaccessible to most of the readers of this Journal it seemed desirable to reproduce them here with slight modifications. In his address before the Paris International Mathematical Congress, Professor Hilbert stated that one of the interesting mathematical problems which await solution is the determination of some one system of independent compatible axioms governing and defining arithmetic concepts. In the discussion of this paper Peano of Italy stated that several Italian mathematicians, viz. Burali-Forti, Padoa, and Pieri, had completely solved this problem in memoirs to which references are given in Vol. 7, of Rivista di Matematica.

It is interesting to observe that all the axioms of the group theory are included in the above axioms of arithmetic. The system of axioms (completely defining and governing the concepts of group theory) which is frequently given, is composed of A 4, 5 and B 3. When the order of the group is finite A 5 may be replaced by the following simpler axiom: From each of the equations

$$ab=ab' \text{ or } ba=b'a$$

it results that $b=b'$. It is thus apparent that the concept of operator in the theory of groups is much less restricted than the concept of real numbers, and includes the latter with respect to multiplication. Numbers have, however, an additional law of combination, viz. addition, while the operators of a group are combined according to a single law of combination generally called multiplication.

LUCUS A NON LUCENDO.

By A. LATHAM BAKER, Ph. D., University of Rochester, Rochester, N. Y.

I reproduce here a curious piece of mathematical legerdemain which proves without proving. It has been, with its variations, the standard for many years in most all the text books. I think it will be difficult to find a proof (?) more utterly lacking in every pedagogical requirement, and yet correct in its final conclusion.

THEOREM. *Any continuous arc of the evolute is equal to the difference between the radii of curvature of the involute that are tangent to the arc at its extremities.*

If r is the radius of curvature and a, b the coördinates of the center of curvature, then the circle of curvature is

$$(x-a)^2+(y-b)^2=r^2 \ldots . (1).$$

But for a normal through a, b we get

$$y-b=-\frac{dx}{dy}(x-a) \ldots . (2),$$

and since the normal is tangent to the evolute

$$y-b=\frac{db}{da}(x-a) \ldots . (3).$$

From (1) and (2), if we suppose the point x, y to move along the curve, and therefore y, a, b, and r to be functions of x, we get

$$(x-a)dx+(y-b)dy-(x-a)da-(y-b)db=rdr,$$

and from (2), $(x-a)dx+(y-b)dy=0$, whence

$$(x-a)da+(y-b)db=rdr \ldots . (4).$$

From (3) and (4), $(x-a)\dfrac{da^2+db^2}{da^2}=-rdr \ldots . (5).$

From (1) and (3), $(x-a)^2\dfrac{da^2+db^2}{da^2}=r^2 \ldots . (6).$

(5) squared, divided by (6), gives $da^2+db^2=dr^2=ds^2$.

Whence $ds=\pm dr \ldots . (7).$ Q. E. D.

Now, to begin with, calling r the radius of curvature does not make it such. We might just as rightfully call it *not* the radius of curvature. Under this latter hypothesis what does equation (7) prove?

To call such a sequence of *disjecta membra* a proof is little short of absurdity, however correct the formal algebraic operations may be. One seldom runs across such a total lack of syllogistic sequence of thought presented under the guise of proof.

Let us see what the r of equation (7) really does represent.

(1) puts a, b at the center of a circle through x, y.

(2) puts *another* a, b on the normal through the locus of another x, y.

(3) puts a third x, y on the tangent to the locus of a, b.

(4)=1st condition + 2d condition. Unites the first and second x, y, and the first and second a, b.

(5)=4th condition+3d condition. puts a, b on the normal to x, y, and at the center of circle r, and *tangent to the normal*, and *therefore* on the evolute, at the center of curvature, and the arbitrary r now becomes the definite R, the radius of curvature, and the a, b becomes α, β, the coördinates of the center of curvature.

(6)=1st condition+3d condition, puts a, b at center of circle through x, y and tangent to the *secant* through x, y, and leaves r and a, b arbitrary.

(7)=5th condition+6th condition, converts the r of (6) into the R of (5), and the a, b into α, β, and hence (7) becomes $\sqrt{(d\alpha^2+d\beta^2)}=dS=\pm dR$, and the theorem follows.

It will be noticed that there is not a suspicion of anything pertaining to the evolute until we reach equation (5), and that then by conjunction of conditions the thing snaps into place and the a, b locus becomes, *willy nilly*, the α, β locus or evolute, and the r similarly becomes R, the radius of curvature. The terminology applied to a, b, r is ineffective. It is the conjunction of conditions that decide what they should be called.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

142. Proposed by M. J. CRAWFORD, Principal of Crawford's Academy, Savannah, Ga.

A gentleman has a garden 400 feet long and 300 feet wide, which he wishes to raise 9 inches higher by means of the earth to be dug out of a ditch 6 feet wide and surrounding the entire garden. How deep must the ditch be?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

$400\times300\times\frac{9}{12}=90000$ cubic feet of earth necessary to raise the garden.

$2(412+300)\times6=8544$ square feet, area of surface of ditch.

$90000\div8544=10\frac{9\,5}{17\,8}$ feet, depth of ditch.

143. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

A's income=a/bth part=$\frac{3}{4}$ of B's income. A's outgo=m/nth part=$\frac{1}{2}$ of B's income. B's outgo=p/qth part=1/1 of A's income. What is the ratio of their savings?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A's outgo=m/nth of B's income.

A's saving=$\left(\frac{a}{b}-\frac{m}{n}\right)=\frac{an-bm}{bn}$ of B's income.

B's outgo=ap/bqth of B's income.

B's saving=$1-\frac{ap}{bq}=\frac{bq-ap}{bq}$ of his income.

$$\therefore \frac{an-bm}{bn}:\frac{bq-ap}{bq}=\frac{an-bm}{n}:\frac{bq-ap}{q},$$

but $a=3$, $b=4$, $m=p=q=1$, $n=2$.

$\therefore$ The ratio of their savings is 1:1.

ALGEBRA.

117. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Rationalize $l^{\frac{1}{4}}+m^{\frac{1}{4}}+n^{\frac{1}{4}}+x^{\frac{1}{4}}+y^{\frac{1}{4}}+z^{\frac{1}{4}}=0$.

Solution by the PROPOSER.

Let $u^6+pu^4+qu^3+ru^2+su+t=0$, be the equation having $l^{\frac{1}{4}}$, $m^{\frac{1}{4}}$, $n^{\frac{1}{4}}$, $x^{\frac{1}{4}}$, $y^{\frac{1}{4}}$, $z^{\frac{1}{4}}$ for its roots; then $u^3+pu^2+qu^{\frac{3}{2}}+ru+su^{\frac{1}{2}}+t=0$ is the equation having l, m, n, x, y, z for its roots.

$$\therefore u^3+pu^2+ru+t+u^{\frac{1}{2}}(qu+s)=0$$

$$\text{or } u^6+2pu^5+(p^2+3r)u^4+(2t+2pr-q^2)u^3$$
$$+(r^2+2pt-2qs)u^2+(2rt-s^2)u+t^2=0.$$

Let $a=l+m+n+x+y+z=\Sigma(l)$.

$b=lm+ln+lx+ly+lz+mn+mx+my+mz+nx+ny+nz+xy+xz+yz=\Sigma(lm)$.

$c=lmn=lmx+lmy+lmz+lnx+lny+lnz+lxy+lxz+lyz+mnx+mny+mnz+mxy+mxz+myz+nxy+nxz+nyz+xyz=(\Sigma(lmn)$.

$d=lmnx+lmny+lmnz+lnxy+lnxz+lxyz+lmxy+lmxz+lmyz+lnyz+lxyz+lmxy+lmxz+lmyz+mnxy+mnxz+mnyz+mxyz+nxyz=\Sigma(lmnx)$.

$e=lmnxy+lmnxz+lmnyz+lnxyz+lmxyz+mnxyz=\Sigma(lmnxy)$.

$f=lmnxyz$.

Then $a=2p$, $b=p^2+2r$, $c=2t+2pr-q^2$, $d=r^2+2pt-2qs$, $e=2rt-s^2$, $f=t^2$.

$\therefore p=\frac{1}{2}a$, $r=\frac{1}{8}(4b-a^2)$, $q^2=\frac{1}{8}(16t+4ab-a^3-8c)$..(1), $s^2=\frac{1}{4}(4bt-a^2t-4e)$..(2).

$qs=(16b^2-8a^2b+a^4+64at-64d)/128$....(3).

$(1)\times(2)=[(3)]^2$.

$\therefore [(16b^2-8a^2b+a^4-64d)/128]^2+(at/128)(16b^2-8a^2b+a^4-64d)+\frac{1}{4}a^2t^2=(64bt^2-16a^2t^2-16abe+4a^3e+32ce)/32+(16ab^2-4a^3b-32bc-4a^3b+a^5+8a^2ct-64e)t/32$.

But $f=t^2$.

$\therefore [(16b^2-8a^2b+a^4-64d)/128]^2+(96a^2f-256bf+64abe-16a^3e-128ce)/128=(48ab^2-24a^3b-256e-128bc+3a^5+32a^2c+64ad)t/128$.

$\therefore \{(16b^2-8a^2b+a^4-64d)^2+[128(96a^2f-256bf+64abe-16a^3e-128ce)]\}^2=[128(48ab^2-24a^3b+3a^5+32a^2c+64ad-128bc-256e)]^2f$.

The values of a, b, c, d, e, f in this last equation is the rationalized form in full.

Also solved by *ELMER SCHUYLER*.

118. Proposed by FREDERIC R. HONEY, Ph. B., Instructor at Trinity College, and Lecturer at Smith College, New Haven, Conn.

An army whose length is equal to a, moves forward. An officer is sent from the rear to the van, and is required to present himself at the rear again when the rear has reached the point where the van was when the army began to move. How far did the officer travel?

I. Solution by H. C. WHITAKER, A. M., Ph. D., Manual Training School, Philadelphia, Pa.; S. F. NORRIS, Baltimore City College, Baltimore, Md.; C. E. ARMENTROUT, Rockingham Military Institute, Mt. Crawford, Va.; J. W. YOUNG, Cornell University, Ithaca, N. Y.; M. E. GRABER, Heidelberg University, Tiffin, Ohio; P. S. BERG, Larimore, N. D.; J. F. TRAVIS, Student Ohio State University, Columbus, O.; MARTIN J. SPINKS, Wilmington, O.; and the PROPOSER.

When the officer moves forward $a+x$, the army goes x; when the officer moves backward x, the army goes $a-x$.

Therefore, $\frac{a+x}{x}=\frac{x}{a-x}$, whence $x=\frac{a}{\sqrt{2}}$.

Whole distance traveled by the officer$=a+2x=a(1+\sqrt{2})$.

II. Solution by C. HORNUNG, A. M., Heidelberg University, Tiffin, O.; D. G. DORRANCE, Jr., Camden, N. Y.; JOSIAH H. DRUMMOND, LL. D., Portland, Me.; and the late SYLVESTER ROBINS, North Branch, N. J.

Let x=officer's rate, and y the army's rate of travel.

Then $\frac{a}{x-y}+\frac{a}{x+y}=\frac{a}{y}$. Whence $x=y(1\pm\sqrt{2})$, or $x/y=1+\sqrt{2}=2.414+$

that is, the officer travels 2.414 times as fast as the army, and, therefore, 2.414 times as far. Therefore the officer travels $(1+\sqrt{2})a$.

Also solved by *J. SCHEFFER, COOPER D. SCHMITT, H. C. WILKES*, and *G. B. M. ZERR*.

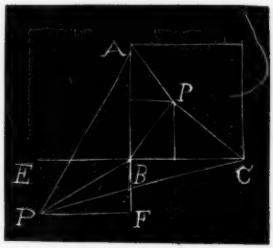
GEOMETRY.

151. Proposed by **FRANK A. GRIFFIN**, Assistant in Mathematics, University of Colorado, Boulder, Col.

A point is at a distance of 1 inch, 2 inches, and $2\frac{1}{2}$ inches, respectively, from three corners of a square. Construct the square. Also solve for the general distances a, b, c.

Solution by **G. B. M. ZERR**, A. M., Ph. D., The Temple College, Philadelphia, Pa.; **P. S. BERG**, B. Sc., Larimore, N. D.; **G. W. DROKE**, State University, Lafayette, Ark.; and **MARTIN SPINKS**, Wilmington, Ohio.

Let P be the point, $PA=a$, $PB=b$, $PC=c$, $AB=x$=side of square, $PE=m$, $PF=n$, $m^2+n^2=b^2$....(1).



$a^2-(x+m)^2=a^2-x^2-2mx-m^2=n^2$....(2).

$c^2-(x+n)^2=c^2-x^2-2nx-n^2=m^2$....(3).

Adding (2) and (3) and using (1) we get

$$n+m=\frac{a^2+c^2-2b^2-2x^2}{2x}\ldots.(4).$$

Subtracting (3) from (2) we get

$$n-m=\frac{c^2-a^2}{2x}\ldots(5). \quad \therefore\ n=\frac{c^2-b^2-x^2}{2x},\quad m=\frac{a^2-b^2-x^2}{2x}.$$

These values of m and n in (1) give

$$x^4-(a^2+c^2)x^2=b^2(a^2+c^2)-\frac{a^4+c^4+2b^4}{2}.$$

$$\therefore\ x=\{\tfrac{1}{2}[a^2+c^2\pm\sqrt{4b^2(a^2+c^2-b^2)-(a^2-c^2)^2}]\}^{\frac{1}{2}}$$

$=1.5163$ inches when P is without the square.

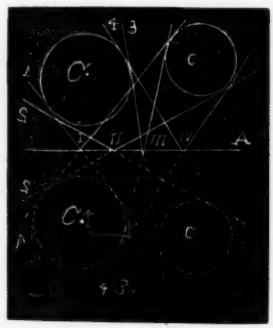
$=2.8197$ inches when P is within the square.

I. When P is without, lay off $PE=m$, perpendicular to PE lay off $EB=n$ and $BC=x$, also AB perpendicular to $EB=x$. The square is now determined.

II. Similarly, when P is within, only $EC=x-n$.

152. Proposed by **ELMER SCHUYLER**, Reading, Pa.

Find a point in a given straight line such than tangents drawn from it to two given circles shall make equal angles with the line.



Solution by **MARCUS BAKER**, Washington, D. C.

In the annexed figure let A be the given line, and C and c the given circles.

Regarding line A as an axis, revolve C and c about it until they fall at C' and c'.

Then draw the four common tangents 1, 2, 3, and 4 of circles C and c' intersecting the axis in I, II, III, and IV. Each of these points fulfils the conditions. The proof is obvious.

Also solved by *G. B. M. ZERR*, and *D. B. NORTHROP*.

CALCULUS.

110. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

Find the volume removed by boring an auger hole through a right circular cone, the radius of the auger being r, the radius of the cone R, and the altitude h, and the axis of the auger intersecting axis of the cone at right angles and at a distance c from the vertex.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $R^2/h^2=a^2$, and suppose the cone to have one nappe.

Then $y^2+z^2=a^2x^2$ is the equation to the cone, and $(x-c)^2+y^2=r^2$ is the equation to the auger.

When $y=ax$, $z=0$, $x_1=\dfrac{c+\sqrt{[r^2(a^2+1)-a^2c^2]}}{a^2+1}$, $x_2=\dfrac{c-\sqrt{[r^2(a^2+1)-a^2c^2]}}{a^2+1}$

Let $\sqrt{[r^2-(x-c)^2]}=y'$, $[r^2-(x-c)^2][a^2x^2-r^2+(x-c)]=A^2$.

I. When $c<r$.

$$V=4\int_{x_1}^{c+r}\int_0^{y'}\sqrt{[a^2x^2-y^2]}\,dxdy+4\int_0^{x_1}\int_0^{ax}\sqrt{[a^2x^2-y^2]}\,dxdy$$

$$=2\int_{x_1}^{r+c}\left(A+a^2x^2\sin^{-1}\frac{y'}{ax}\right)dx+\pi a^2\int_0^{x_1}x^2dx$$

$$=2\int_{x_1}^{r+c}\left(A+\frac{a^2x^2(cx+r^2-c^2)}{3A}\right)dx+\tfrac{1}{3}\pi a^2x_1^3=2\int_{x_1}^{r+c}Bdx+\tfrac{1}{3}\pi a^2x_1^3.$$

II. $c>r$, but $r<ac/(a^2+1)$.

$$V=4\int_{c-r}^{c+r}\int_0^{y'}\sqrt{[a^2x^2-y^2]}\,dxdy=2\int_{c-r}^{c+r}Bdx.$$

III. $c>r$, but $r>ac/(a^2+1)$.

$$V=4\int_{x_1}^{c+r}\int_0^{y'}\sqrt{[a^2x^2-y^2]}\,dxdy+4\int_{c-r}^{x_2}\int_0^{y'}\sqrt{[a^2x^2-y^2]}\,dxdy$$

$$+4\int_{x_2}^{x_1}\int_0^{ax}\sqrt{[a^2x^2-y^2]}\,dxdy=2\int_{x_1}^{c+r}Bdx+2\int_{c-r}^{x_2}Bdx+\tfrac{1}{3}\pi a^2(x_1^3-x_2^3).$$

Let $u=x-c$ and $l^2=R^2+h^2$.

$$\therefore A=\frac{1}{hl}\sqrt{\{(r^2-u^2)[(l^2u+R^2c)^2+h^2(R^2c^2-l^2r^2)]\}}.$$

Let $h^2(R^2c^2-l^2r^2)=b^2l^4$, $R^2c/l^2=d$.

Then $A=(l/h)\sqrt{\{(r^2-u^2)[b^2+(u+d)^2]\}}$.

This can be integrated by a process similar to that employed by Professor Finkel in Vol. V, No. 1, pages 20, 21.

Let $Rc-lr$, then $A=\frac{1}{h}(lu+Rr)\sqrt{(r^2-u^2)}=\frac{R}{rh}(cu+r^2)\sqrt{(r^2-u^2)}$.

$$\therefore V=\frac{2R}{hr}\int_{-r}^{r}(cu+r^2)\sqrt{(r^2-u^2)}du+\frac{2Rr}{3h}\int_{-r}\sqrt{\frac{(u+c)^2du}{(r^2-u^2)}}$$

$$=\frac{2\pi Rr}{3h}(2r^2+c^2).$$

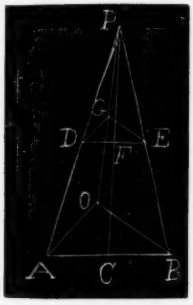
111. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

(*a*). Find the dimensions of a cup, capacity c, in the form of a frustum of a regular pyramid of n faces, so that its internal surface is a minimum.

(*b*). Find the dimensions of a cup, capacity c, in the form of a frustum of a hyperbolioid or of a paraboloid, whichever it is, so that its internal surface is a minimum.

Solution by the PROPOSER.

(*a*). Let $PO=x$, $PG=y$. $\angle AOB=\angle DGE=2\pi/n$, $AO=r$, where O is the center of the circle circumscribing the larger base and G the center of the circle circumscribing the smaller base.



Then $DG=ry/x$, $AB=2r\sin(\pi/n)$, $DE=(2ry/x)\sin(\pi/n)$. Area $DGE=(r^2y^2/x^2)\sin(\pi/n)\cos(\pi/n)$.

$\therefore$ Area of upper base$=(nr^2y^2/x^2\sin(\pi/n)\cos(\pi/n)$.

$PC=\sqrt{(PA^2-AC^2)}=\sqrt{[x^2+r^2-r^2\sin^2(\pi/n)}$

$=\sqrt{[x^2+r^2\cos^2(\pi/n)}$.

Similarly $DF=(y/x)\sqrt{[x^2+r^2\cos^2(\pi/n)]}$.

Area $ADEB=r\sin(\pi/n)\sqrt{[x^2+r^2\cos^2(\pi/n)][(x^2-y^2)/x^2]}$.

The surface is the least when the upper or smaller base is the bottom of the cup.

Total surface of cup$=u$, volume$=c$.

$$\therefore u=nr\sin(\pi/n)\sqrt{[x^2+r^2\cos^2(\pi/n)]}\left(\frac{x^2-y^2}{x^2}\right)+\frac{nr^2y^2}{x^2}\sin(\pi/n)\cos(\pi/n)$$

$$c=\tfrac{1}{3}nr^2\sin(\pi/n)\cos(\pi/n)\left(\frac{x^3-y^3}{x^2}\right).\quad\text{Let } (r/x)\cos(\pi/n)=\tan\theta.$$

$$\therefore u=n\tan\theta\sec\theta\sec(\pi/n)\sin(\pi/n)(x^2-y^2)+ny^2\tan^2\theta\sec(\pi/n)\sin(\pi/n).\ (1).$$

$$c=\tfrac{1}{3}n\tan^2\theta\sec(\pi/n)\sin(\pi/n)(x^3-y^3)\ldots.(2).$$

Differentiating (1) and (2) we get

$$\frac{dx}{dy}=\left(\frac{\sec\theta-\tan\theta}{\sec\theta}\right)\frac{y}{x}=(1-\sin\theta)\,\frac{y}{x}\ldots.(3). \qquad \frac{dx}{dy}=\frac{y^2}{x^2}\ldots.(4).$$

$$\frac{dx}{d\theta}=-\frac{(\sec^2\theta+\tan^2\theta)(x^2-y^2)+2y^2\sec\theta\tan\theta}{2x\tan\theta}\ldots.(5).$$

$$\frac{dx}{d\theta}=-\frac{2\sec^2\theta(x^3-y^3)}{3x^2\tan\theta}\ldots.(6).$$

From (3) and (4) $y=x(1-\sin\theta)\ldots.(7)$.

From (5) and (6), $\sec^2\theta+\tan^2\theta)(x^2-y^2)+2y^2\sec\theta\tan\theta=\frac{4\sec^2\theta(x^3-y^3)}{3x}$

This reduces to $(1+\sin^2\theta)(x^2-y^2)+2y^2\sin\theta=4(x^3-y^3)/3x\ldots.(8)$.

(7) in (8) gives $(3-8\sin\theta+3\sin^2\theta)\sin\theta=0$.
$\therefore$ $\sin\theta=0$ or $\frac{1}{3}(4\pm\sqrt{7})$.
$\sin\theta$ cannot be zero nor greater than unity.
$\therefore$ $\sin\theta=\frac{1}{3}(4-\sqrt{7})\ldots.(9)$.

$\therefore$ $\tan\theta=\sqrt{\frac{4\sqrt{7}-7}{14}}$. $\ldots(10)$.

From (7) and (9), $3y=x(\sqrt{7}-1)\ldots.(11)$.

(10) and (11) in (2) gives $x=\frac{1}{3}\left(\frac{2(38\sqrt{7}+89)c}{n\tan(\pi/n)}\right)^{\frac{1}{3}}$.

From (11), $y=\frac{1}{3}x(\sqrt{7}-1)=\frac{1}{3}\left(\frac{4(\sqrt{7}+13)c}{n\tan(\pi/n)}\right)^{\frac{1}{3}}$.

$x-y=x-\frac{1}{3}x(\sqrt{7}-1)=\frac{1}{3}x(4-\sqrt{7})=\left(\frac{2(\sqrt{7}-2)c}{n\tan(\pi/n)}\right)^{\frac{1}{3}}$=altitude.

$r=x\tan\theta\sec(\pi/n)=\left(\frac{(91+88\sqrt{7})c^2}{98n^2\tan^2(\pi/n)}\right)^{\frac{1}{3}}\sec(\pi/n)$.

$AB=2r\sin(\pi/n)=2\left(\frac{(91+88\sqrt{7})c^2}{98n^2\tan^2(\pi/n)}\right)^{\frac{1}{3}}\tan(\pi/n)$, side lower base.

$DE=\frac{2ry}{x}\sin(\pi/n)=2\left(\frac{4(11\sqrt{7}-28)c^2}{49n^2\tan^2(\pi/n)}\right)^{\frac{1}{3}}\tan(\pi/n)$, side upper base.

(2). Let $y^2=4ax$ be the equation to the parabola, then we get from the Integral Calculus the two equations, between the limits x_2 and x_1,

$$u=\tfrac{8}{3}\pi\sqrt{a}[(x_2+a)^{\frac{3}{2}}-(x_1+a)^{\frac{3}{2}}]+4\pi a x_1\ldots.(1).$$
$$c=2\pi a(x_2^2-x_1^2)\ldots.(2).$$

From (1) and (2), by differentiation, we get

$$\frac{dx_2}{dx_1}=\frac{(x_1+a)^{\frac{1}{2}}-\sqrt{a}}{(x_2+a)^{\frac{1}{2}}}\ldots.(3);\quad \frac{dx_2}{dx_1}=\frac{x_1}{x_2}\ldots.(4).$$

$$\frac{dx_2}{da}=-\frac{(x_2+a)^{\frac{3}{2}}-(x_1+a)^{\frac{3}{2}}+3a(x_2+a)^{\frac{1}{2}}-3a(x_1+a)^{\frac{1}{2}}+3\sqrt{a}\,x_1}{3a(x_2+a)^{\frac{1}{2}}}\ldots.(5).$$

$$\frac{dx_2}{da}=-\frac{x_2^2-x_1^2}{2ax_2}\ldots.(6).$$

From (3) and (4) by eliminating dx_2/dx_1 we get

$$x_1=\frac{x_2^2-2x_2\sqrt{[a(x_2+a)]}}{x_2+a}\text{ and }x_1=0\ldots.(7).$$

Eliminating dx_2/da between (5) and (6) and substituting the first value of x_1 from (7) in the resulting equation, we get after reduction

$$36x_2^3-100ax_2^2-279a^2x_2-144a^3=0\ldots\ (8).$$

Let $a/x_2=z$ and (8) becomes

$$144z^3+279z^2+100z-36=0\ \ldots(9).$$

Let $z=v-\frac{31}{48}$ and (9) becomes

$$v^3-\frac{12}{23}\frac{28}{04}\frac{83}{}v-\frac{88}{55}\frac{33}{29}\frac{3}{6}=0\ldots.(10).$$

$\therefore\ v_1=.861506,\ z_1=.215673.$

$v_2=-.445379,\ z_2=1.094212.$

$v_3=-.416138,\ z_3=-1.061971.$

$\therefore\ a=.215673x_2,\ -1.094212x_2,$ or $-1.061971x_2.$

a cannot be negative. The first value of a gives $x_1=-.019813x_2$, a negative value and therefore not admissible. From this we learn that the second value of x_1 in (7), $x_1=0$ is the possible value and the cup is not a frustum of a paraboloid, but a paraboloid, a cup with a curved bottom.

$x_1=0$ in (5) and (6) $x_2^2-15ax_2+48a^2=0.$

$\therefore\ x_2=\frac{15\pm\sqrt{33}}{2}a.\quad\therefore\ x_2=\frac{15-\sqrt{33}}{2}a\ldots(11)$ is the admissible value.

From (11) and (2), $x_2=\left(\frac{(15-\sqrt{33})c}{4\pi}\right)^{\frac{1}{4}}$=altitude of cup.

$$a=\left(\frac{(43+5\sqrt{33})c}{3072\pi}\right)^{\frac{1}{4}}=\left(\frac{2c}{\pi(15-\sqrt{33})^2}\right)^{\frac{1}{4}}.$$

$y_2=2\sqrt{a}x_2=2\left(\frac{(15+\sqrt{33})c^2}{384\pi^2}\right)^{\frac{1}{8}}$, radius of top.

$$u=\tfrac{1}{96}\left(\frac{(1337+215\sqrt{33})4\pi c^2}{9}\right)^{\frac{1}{4}}[(3298-450\sqrt{33})^{\frac{1}{4}}-2].$$

MECHANICS.

116. Proposed by C. L. CHILTON, Greensboro, Ala.

Given, the shaft ABC attached at one end by a pivot to the piston-rod of an engine (at A) and the other to a crank of the wheel CDE (at C). The shaft moves through the distance of two feet in one second from A to B and at the same time turns the crank from C to E. The force propelling the shaft along the constrained course from A to B is 5760 pounds. The mass of the rod and wheel and friction being not considered, what would be the kinetic energy of the machine? or the sum of the moment around O, the center of the wheel?

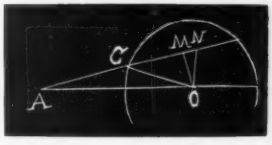
Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let AC, the connecting rod$=l$, $OC=r=$one foot, $\angle AOC=\theta$, force 5760 pounds along $AO=P$, component of P along $AC=Q$. Then moment of crank effect about $O=Q.OM$. In the right triangles AOM, AON, $AO:OM=AN:ON$.

$\therefore P:OM=Q:ON.$

$\therefore Q.OM=P.ON.$

Also $ON:OC=\sin OCN:\sin ONC$.



Let $\angle OAC=\varphi$. Then $\angle ONC=\frac{1}{2}\pi-\varphi$, $\angle OCN=\theta+\varphi$.

$$\therefore ON=\frac{r\sin(\theta+\varphi)}{\cos\varphi}=r\sin\theta+\frac{r\cos\theta\sin\varphi}{\cos\varphi}.$$

$$\text{but } \sin\varphi=\frac{r\sin\theta}{l}. \quad \therefore ON=r\sin\theta+\frac{r^2\sin\theta\cos\theta}{\sqrt{(l^2-r^2\sin^2\theta)}}.$$

$$\therefore \text{moment}=Pr\sin\theta\left(1+\frac{r\cos\theta}{\sqrt{(l^2-r^2\sin^2\theta)}}\right).$$

Now θ varies from 0 to π.

$$\therefore \text{Average moment}=\frac{Pr\int_0^{\pi}\sin\theta\left(1+\frac{r\cos\theta}{\sqrt{(l^2-r^2\sin^2\theta)}}\right)d\theta}{\int_0^{\pi}d\theta}=\frac{2Pr}{\pi},$$

a result independent of the connecting rod.

$2Pr/\pi=.6366Pr=3666.816$ or 3667 pounds.

117. Proposed by F. P. MATZ. M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

How much lower must *one end* of a heavy uniform chain, wound round the circumference of a perfectly rough vertical wheel, hang than *the other end*, when the chain is on the point of motion?

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

Let l=the entire length of the chain, a=the radius of the wheel, x=the longest part hanging down, $l-(\pi a+x)$=the shortest part, θ=the angle which the radius through any point of the string makes with the vertical diameter and positive in the direction left to right, T and T' the tensions at any points in those parts of the string to which x and $l-(\pi a+x)$ belong, and μ=the coefficient of friction.

We now have, from the usual theory, a unit of length of the chain being taken as a unit of weight.

$$T=Ce^{\mu\theta}+\frac{a}{1+\mu^2}[2\mu\sin\theta+(1-\mu^2)\cos\theta]\ldots.(1).$$

When $\theta=\frac{1}{2}\pi$, $T=x$, and (1) is

$$x=Ca^{\frac{1}{2}(\mu\pi)}+\frac{2\mu a}{1+\mu^2},\text{ or }C=\left(x-\frac{2\mu a}{1+\mu^2}\right)e^{-\frac{1}{2}(\mu\pi)}\ldots.(2),$$

and (1) is $T=\left(x-\frac{2\mu a}{1+\mu^2}\right)e^{\mu(\theta-\frac{1}{2}\pi)}+\frac{a}{1+\mu^2}[-2\mu\sin\theta+(1-\mu^2)\cos\theta]\ldots.(3).$

In like manner, $T'=l-(\pi a+x)$ when $\theta=-\frac{1}{2}\pi$, and

$$T'=\left(l-\pi a-x+\frac{2\mu a}{1+\mu^2}\right)e^{\mu(\theta+\frac{1}{2}\pi)}+\frac{a}{1+\mu^2}[-2\mu\sin\theta+(1-\mu^2)\cos\theta]\ldots.(4).$$

For equilibrium, $T=T'$ at the vertex, where $\theta=0$.

$$\therefore\left(l-\pi a-x+\frac{2\mu a}{1+\mu^2}\right)e^{\frac{1}{2}(\mu\pi)}=\left(x-\frac{2\mu a}{1+\mu^2}\right)e^{-\frac{1}{2}(\mu\pi)}\ldots.(5),$$

giving $x=\frac{l-\pi a}{1+e^{-\mu\pi}}+\frac{2\mu a}{1+\mu^2}\ldots.(6).$

Then $2x+\pi a-l$ is found, the required length.

Also solved by *G. B. M. ZERR.*

AVERAGE AND PROBABILITY.

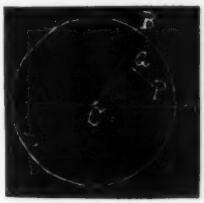
101. Proposed by L. C. WALKER, Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

By direct calculation obtain the average distance between two points in the surface of a circle.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let P, Q be the random points in the sector AOB. $AO=r$, $OP=y$, $OQ=x$,

$\angle AOB=\beta$, $\angle AOQ=\theta$, $\angle AOP=\varphi$. An element of the sector at P is $ydyd\varphi$; at Q, $xdxd\theta$. The limits of x, are 0 and r; of y, 0 and x; of θ, 0 and β; of φ, 0 and θ.



$$PQ=u=[x^2+y^2-2xy\cos(\theta-\varphi)]^{\frac{1}{2}}.$$

$$\therefore \triangle=\frac{\int_0^\beta\int_0^\theta\int_0^r\int_0^x uxdxydyd\theta d\varphi}{\int_0^\beta\int_0^\theta\int_0^r\int_0^x xdxyd\theta d\varphi}$$

$$=\frac{16}{\beta^2r^4}\int_0^\beta\int_0^\theta\int_0^r\int_0^x uxyd\theta d\varphi dxdy=\frac{8}{3\beta^2r^4}\int_0^\beta\int_0^\theta\int_0^r\{16\sin^3\tfrac{1}{2}(\theta-\varphi)$$

$$+12\sin^3\tfrac{1}{2}(\theta-\varphi)\cos(\theta-\varphi)-2+3\cos^2(\theta-\varphi)$$
$$+3\sin^2(\theta-\varphi)\cos(\theta-\phi)\log[1+\operatorname{cosec}\tfrac{1}{2}(\theta-\varphi)]\}x^4d\theta d\varphi dx$$

$$=\frac{8r}{15\beta^2}\int_0^\beta\int_0^\theta\{16\sin^3\tfrac{1}{2}(\theta-\varphi)+12\sin^3\tfrac{1}{2}(\theta-\varphi)\cos(\theta-\varphi)-2+3\cos^2(\theta-\varphi)$$

$$+4\sin^2(\theta-\varphi)\cos(\theta-\varphi)\log[1+\operatorname{cosec}\tfrac{1}{2}(\theta-\varphi)]\}d\theta d\varphi$$

$$=\frac{4r}{45\beta^2}\int_0^\beta\{48\sin^4\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta+12\sin^3\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-32\sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-6\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta$$

$$-64\cos\tfrac{1}{2}\theta+64+9\sin\theta\cos\theta+6\sin^3\theta\log(1+\operatorname{cosec}\theta)\}d\theta$$

$$=\frac{2r}{135\beta^2}\{16\sin^6\tfrac{1}{2}\beta+16\cos^6\tfrac{1}{2}\beta+96\sin^5\tfrac{1}{2}\beta+24\sin^4\tfrac{1}{2}\beta-12\cos^4\tfrac{1}{2}\beta-112\sin^3\tfrac{1}{2}\beta$$

$$-36\sin^2\tfrac{1}{2}\beta-720\sin\tfrac{1}{2}\beta+27\sin^2\beta+384\beta-4$$
$$-12(\sin^2\beta\cos\beta+2\cos\beta+2)\log(1+\sin\tfrac{1}{2}\beta)+12(\sin^2\beta\cos\beta$$
$$+2\cos\beta-2)\log\sin\tfrac{1}{2}\beta\}.$$

For the circle, $\beta=2\pi$, $\triangle=\dfrac{128r}{45\pi}$.

For the semicircle, $\beta=\pi$, $\triangle=\dfrac{256r}{45\pi}-\dfrac{1472r}{135\pi^2}$.

For the quadrant, $\beta=\frac{1}{2}\pi$.

$$\triangle=\frac{32r}{135\pi}\left(48+\frac{3}{\pi}-\frac{94\sqrt{2}}{\pi}-\frac{6}{\pi}\log\frac{1+\sqrt{2}}{2}\right).$$

II. Solution by the PROPOSER.

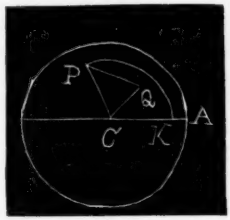
Take the center, C, of the circle and the horizontal radius CA as pole and initial line in polar coördinates.

Let $P(x, \theta)$ and $Q(y, \phi)$ be the random points. Then we have

$$PQ = \sqrt{x^2+y^2-2xy\cos(\theta-\phi)}.$$

Now with C as a center and a radius CP describe the arc PK.

It is evidently necessary to consider only those positions of the two points in which Q is confined to the sector PCK, since under this limitation PQ will take all possible distances.



An element of area of the circle at the point P is $xdxd\theta$, and at the point Q it is $ydyd\phi$.

The limits of y are 0 and x; of x, 0 and r; of ϕ, 0 and θ; and those of θ, 0 and 2π.

Consequently, we have for the required mean distance

$$M = \frac{1}{(\pi r^2)^2}\int_0^{2\pi}\int_0^{\theta}\int_0^{r}\int_0^{x}\sqrt{[x^2+y^2-2xy\cos(\theta-\phi)]}\,xdxd\theta.ydyd\phi$$

$$= \frac{4}{3\pi^2 r^4}\int_0^{2\pi}\int_0^{\theta}\int_0^{x}\Big[8\sin^3\tfrac{1}{2}(\theta-\phi)+6\sin^3\tfrac{1}{2}(\theta-\phi)\cos(\theta-\phi)$$

$$-\tfrac{3}{2}\sin^2(\theta-\phi)\cos(\theta-\phi)\log\left(\frac{1+\sin\frac{1}{2}(\theta-\phi)}{\sin\frac{1}{2}(\theta-\phi)}\right)-1+\cos^2(\theta-\phi)\Big]x^4\,d\theta d\phi dx$$

$$= \frac{r}{15\pi^2}\int_0^{2\pi}\int_0^{\theta}\Big[8\sin\tfrac{1}{2}(\theta-\phi)+40\sin\tfrac{1}{2}(\theta-\phi)\cos^2\tfrac{1}{2}(\theta-\phi)$$

$$+6\sin^2(\theta-\phi)\cos(\theta-\phi)\log\left(\frac{1+\sin\frac{1}{2}(\theta-\phi)}{\sin\frac{1}{2}(\theta-\phi)}\right)$$

$$-48\sin\tfrac{1}{2}(\theta-\phi)\cos^4\tfrac{1}{2}(\theta-\phi)+3\cos 2(\theta-\phi)-1\Big]d\theta d\phi$$

$$= \frac{2r}{45\pi^2}\int_0^{2\pi}\Big[32-32\cos\tfrac{1}{2}\theta-16\sin^2\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta+24\sin^4\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta+3\sin\theta\cos\theta$$

$$+3\sin^3\theta\log\left(\frac{1+\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\right)\Big]d\theta = \frac{128r}{45\pi}.$$

Professor Walker also furnished a second very excellent solution.

MISCELLANEOUS.

91. Proposed by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

The following sides and area are given for a rational triangle in the table of rational scalene triangles on page 167 of Dr. Halsted's "Metrical Geometry (Boston, 1881), viz.: sides, 21, 61, 65; area, 420. The same sides and area are given in Septimus Tebay's "Mensuration" (London and Cambridge, 1868), in a table on page 113.

The sides of this triangle can not all be correct because they are all *odd*.

Assuming that the *area* given is correct, it is required to determine the error in the sides.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Knowing that every integral rational scalene triangle is the sum or the difference of two integral right triangles having equal altitudes, and knowing that 65 and 61 are hypotenuses of integral right triangles, we find that 65, 63, 16; 65, 60, 25; 65, 56, 33; and 65, 52, 39, are the respective sides of the right triangles having the hypothenuse 65; and that 61, 60, 11, are the sides of the only integral right triangle having the hypothenuse 61.

We observe that the triangles whose respective sides are 65, 60, 25, and 61, 60, 11, have the common leg 60. Taking 60 as the altitude, and finding the *sum* and the *difference* of the two triangles, we obtain two integral rational scalene triangles, sides being 65, 61, 36, and 65, 61, 14. The area of the former is $\frac{1}{2}\sqrt{(81\times16\times20\times45)}=1080$, and of the latter, $\frac{1}{2}\sqrt{(70\times5\times9\times56)}=420$.

$\therefore$ The side of the triangle in question should be 14.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Supposing the sides 61 and 65 to be right, but 21 to be wrong, we may put this side$=2a$, since it must be an even number. Thus, we get $(63+a)(63-a)(a+2)(a-2)=420^2$, whence $a^4-3973a^2=-192276$.

$\therefore\ a=\sqrt{\frac{3973\pm3975}{2}}$. Taking the lower sign, we obtain $2a=14$. Thus, the sides are 14, 61, 65. There may possibly be more rational values for the three sides to produce the area 420.

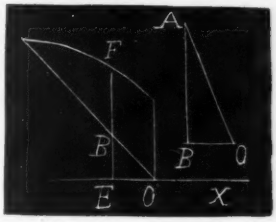
Also solved by G. B. M. ZERR.

92. Proposed by J. T. COLE, Columbus, O.

A staff $a=60$ feet high, casting a shadow on a horizantal plane due north $b=20$ feet long, falls due northeast. Find the area covered by the shadow.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let O, B be the same points in both figures. OC the staff after it has fallen to the ground. OA the staff while falling.



Let $\angle AOB=\theta$, altitude of the sun$=\beta$, $OE=x$, $FE=y$. Then $\beta=\tan^{-1}(a/b)=\tan^{-1}(3)$, $AB=a\sin\theta$, $OB=a\cos\theta$, $OE=EB=(a/\sqrt{2})\cos\theta$, $BF=AB\cot\beta$.

$\therefore\ BF=a\sin\theta\cot\beta=b\sin\theta$.

$\therefore\ x=(a/\sqrt{2})\cos\theta$,

$y=x+b\sin\theta=x+(b/a)\sqrt{(a^2-2x^2)}=y'$.

Area passed over by the shadow$=CODFC$.

$$\therefore\ A=\int_0^{a/\sqrt{2}}\int_x^{y'}dxdy=b/a\int_0^{a/\sqrt{2}}\sqrt{(a^2-2x^2)}dx. \quad \therefore\ A=\tfrac{1}{8}\pi ab\sqrt{2}=150\pi\sqrt{2}.$$

PROBLEMS FOR SOLUTION.

ALGEBRA.

139. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Solve neatly $\sqrt[4]{(m-x)}=\sqrt[4]{n}-\sqrt[4]{x}$.

140. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A man pays monthly \$24.50 for 8 years for a loan of \$1250. What is the rate %?

142. Proposed by JOSEPH V. COLLINS, Ph. D., Professor of Mathematics, State Normal School, Stevens Point, Wis.

How many teams of two horses each can a livery stable man send out who has 10 horses, assuming (1) that we consider the way the team is hitched and (2) that we do not.

Suppose he has 8 horses; 10 horses. Suppose he has 7 buggies, then how many different rigs can he send out, assuming that he has 10 horses, and counting both one and two horse rigs?

[We will publish a series of problems of this character in future numbers of the MONTHLY.]

*** Solutions of these problems should be sent to J. M. Colaw not later than August 10.

GEOMETRY.

169. Proposed by S. F. NORRIS, Professor of Astronomy and Mathematics, Baltimore City College, Baltimore, Md.

Theorem. Two quadrilaterals having three sides of the one equal to the three corresponding sides of the other, each to each, and the two corresponding angles adjacent to the unknown sides equal, each to each, are equal figures. [From *Olney's Geometry*, Section VIII, Proposition XIV.]

1. Required the proof. 2. Is this proposition found in any other text-book of Geometry?

170. Proposed by CHARLES C. CROSS, Whaleyville, Va.

If p, q, r are the distances of the orthocenter from the sides, prove that

$$4\left(\frac{a}{p}+\frac{b}{q}+\frac{c}{r}\right)=\left(\frac{a}{p}+\frac{b}{q}-\frac{c}{r}\right)\left(\frac{b}{q}+\frac{c}{r}-\frac{a}{p}\right)\left(\frac{c}{r}+\frac{a}{p}-\frac{b}{q}\right).$$

171. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the nearest distance of the parabola $y^2=16x$ and the ellipse $16x^2+9y^2-160x-144y+832=0$.

*** Solutions of these problems should be sent to B. F. Finkel not later than August 10.

CALCULUS.

133. Proposed by NELSON L. RORAY, South Jersey Institute, Bridgeton, N. J.

Integrate $\int\frac{\sqrt{(1+y)}}{1+y^2}dy$.

134. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

To find the curve for which the sum of that part of the tangent, lying between the point of contact and the axis of abscissas, and the corresponding ordinate is constant $=c$, and which passes through the point (a, b).

*** Solutions of these problems should be sent to J. M. Colaw not later than August 10.

MECHANICS.

124. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

At one end of a weightless thread of length l is fastened a sphere of radius r, and the other end of the thread is fastened to a vertical axis. The axis is put into motion of constant angular velocity ϕ. What is the maximum angle which the thread will make with the vertical axis?

*** Solutions of this problem should be sent to B. F. Finkel not later than August 10.

DIOPHANTINE ANALYSIS.

87. Proposed by L. C. WALKER, Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Find three numbers in arithmetical progression the sum of whose cubes is a cube.

88. Proposed by L. C. WALKER, Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Find three square numbers in harmonical progression.

*** Solutions of these problems should be sent to J. M. Colaw not later than August 10.

AVERAGE AND PROBABILITY.

110. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find the average area of the triangle formed by joining three random points taken on the surface of a regular hexagon, two on one side of a diagonal and the third on the other side.

111. Proposed by L. C. WALKER, Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

If a radius be drawn at random in a given semi-circle, and a point taken at random in one of the sectors formed, show that the chance that a random line drawn through the point will cut the arc of the sector is

$$1-\frac{1}{\pi^2}\log 2.$$

112. Proposed by L. C. WALKER, Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

Two circles are drawn at random, both in magnitude and position, but so as to lie wholly upon the surface of a given circle. Show that the chance of their both resting on the same diameter of the given circle is

$$\frac{4}{\pi}(8\log 2-5).$$

*** Solutions of these problems should be sent to B. F. Finkel not later than August 10.

MISCELLANEOUS.

110. Proposed by E. W. MORRELL, South Tunbridge, Vt.

If a and b be the sides of a triangle, A and B the angles opposite, then will $\log b - \log a = \cos 2A - \cos 2B + \frac{1}{2}(\cos 4A - \cos 4B) + \frac{1}{3}(\cos 6A - \cos 6B) + \ldots$.

111. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Exhibit $\cos^3\theta \sin^3\theta \sin^2\phi \cos\phi$ as a series of harmonics.

112. Proposed by LON C. WALKER, A. M., Assistant Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

(a) Find the area enclosed by four circles, of which two touch the x-axis, and two the y-axis, at the origin.

(b) Required the area enclosed by four parabolas, of which two touch the x-axis, and two the y-axis, at the origin.

113. Proposed by F. P. MATZ. M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

Deduce the Sylvestrian Reciprocant from $x^4 + y^4 = 4x^2y^2$.

*** Solutions of these problems should be sent to J. M. Colaw not later than August 10.

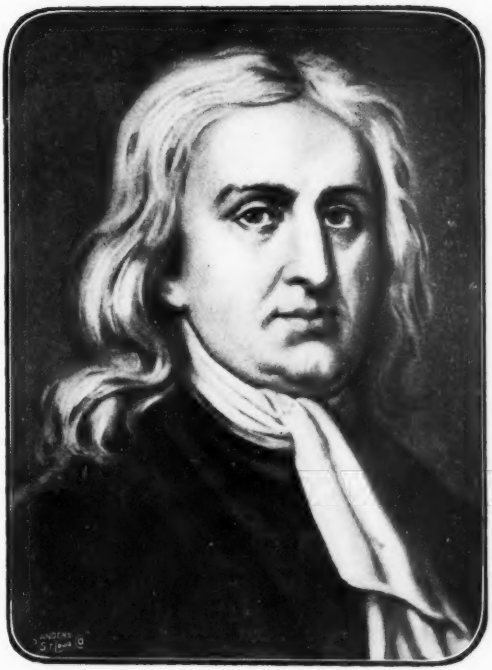
NOTES.

Dr. W. B. Fite has been appointed Instructor in Mathematics at Cornell University. He will take most of the advanced work formerly done by Dr. G. A. Miller.

Professor Alexander S. Chessin has been elected Professor of Mathematics in the Washington Uuiversity, St. Louis. The University is to be congratulated in securing such an able man.

Mr. Paul A. Towne, A. M., has sent us a number of very interesting magic squares constructed on the sides and hypothenuse of right triangles, but owing to the difficulty in reproducing them, we must omit their publication.

Brief notice will be made in the next issue of the MONTHLY of the following books: *One Hundred Problems in Mathematical Physics*, by E. P. Thompson; *Atoms and Energy*, by D. A. Murray; *Experimental Physics*, by Eugene Lommel; *An Introduction to the Study of Chemistry*, by Ira Remsen.



ISAAC NEWTON.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. VIII. AUGUST-SEPTEMBER, 1901. NOS. 8-9.

BIOGRAPHY.

ISAAC NEWTON.

By VIRGINIA J. CRAIG, A. B., Springfield, Mo.

Sir Isaac Newton was born at Woolthorpe in Lincolnshire in 1642, the year of Galileo's death. He first attended the village school and later the public school at Grantham. He was a delicate child and at first far from industrious. An unprovoked attack from a boy above him led to a fight in which Newton's pluck was victorious. This success led him to greater exertions in school, and after a time he rose to be head boy. He early displayed a taste for mechanical inventions. He made wind-mills, water-clocks, kites, dials, and a carriage propelled by the rider. When he had attained his fifteenth year, his mother took him home to assist her in the management of the farm, but his dislike for farming and desire for study induced her to send him back to Grantham, where he remained until his eighteenth year, when he entered Trinity college, Cambridge. Little is known as to his attainments at this time. He tells us that he had bought a book on astrology at a fair, but on account of his ignorance of trigonometry could not understand its figures. So he bought a Euclid, but on looking it over, thought the propositions self evident and laid it aside as a trifling work.

He now applied himself to Descartes' Geometry, which he mastered without assistance. This work inspired him with a love for the higher mathematics. He tells us that in 1665 he found the method of Infinite Series and computed